An Investigation with Logical Rigor of the Problem of the Area of a Triangle on a Cartesian Plane with One Vertex at the Origin*

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November 12, 2021

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*This is my response to an exercise question from the textbook *Number Theory* by George E. Andrews.

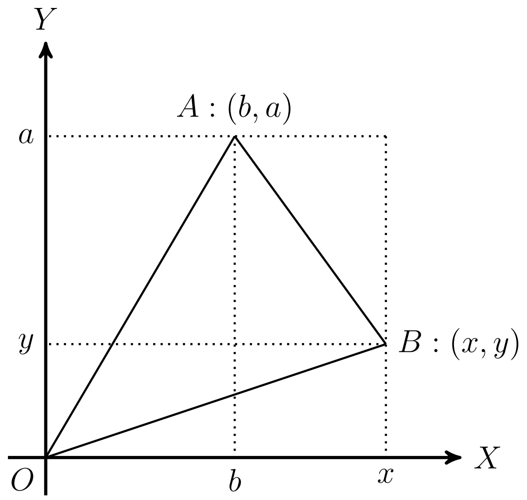


Figure 1: The vertices (b, a) and (x, y) are both in Quadrant I, with $b < x$ and $y < a$.

1 The Problem

Prove that the area of the triangle whose vertices are $(0, 0)$, (b, a) and (x, y) is

$$|by - ax|/2. \tag{1}$$

1.1 Remark

Two vertices (b, a) and (x, y) of the triangle are free, which means that no matter where they are on the Cartesian plane: both in Quadrant I, or one in Quadrant II and the other on the positive half of the X-axis, or whatever, Formula 1 can always be used for its area.

2 Some Arbitrary Cases

We can start with analyzing some arbitrary cases to see if the formulae for them agree with Formula 1. There is a proof only if there is such a preliminary agreement.

The area of $\triangle OAB$ in Figure 1 is the area of the enclosing rectangle minus the area of the three right triangles, which is:

$$ax - ab/2 - xy/2 - (x - b)(a - y)/2 = (ax - by)/2. \tag{2}$$

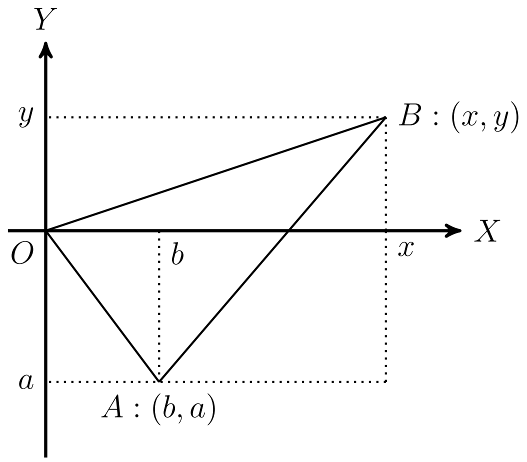


Figure 2: The vertex (b, a) is in Quadrant IV, while (x, y) is in Quadrant I, with $b < x$.

The area of $\triangle OAB$ in Figure 2 is again the area of the enclosing rectangle minus the area of the three right triangles, which is:

$$(y - a)x + ab/2 - xy/2 - (x - b)(y - a)/2 = (by - ax)/2. \tag{3}$$

2.1 Discussion

The distribution of the points (b, a) and (x, y) with respect to the quadrants has many possibilities. Above we explored two cases, and see that the formulae only differ by a sign. This implies that the formula in one case can be used in the other case, provided we change the sign of the result. A plausible albeit not yet formally proved guess is that we can take the formula for one case, and apply it in all cases, and take the absolute value of the formula's yield as the final result. Hence Formula 1.

2.2 Move on !

If I was given this problem when I was a secondary school student, at this point I would have been quite satisfied and ready to accept Formula 1 as the correct formula for all possible cases without bothering to consider what exactly other cases are. But now I am much older and I am studying number theory, which favors logical rigor over taking seemingly natural things for granted, it is worth to try to go a bit further than I would in secondary school. After all, isn't it curious that no matter where you put the two free vertices of the triangle on the Cartesian plane, the formula for the area of

	$y > a$	$y < a$	$y = a$
$x > b$	case 1	case 2	case 7
$x < b$	case 3	case 4	case 8
$x = b$	case 5	case 6	$\times$

Table 1: The eight possible cases when all coordinates are positive. The cross ($\times$) denotes the invalid case where the two vertices coincide and make no triangle.

this triangle, as derived by subtracting triangles from a rectangle, is always either $(by - ax)/2$ or $(ax - by)/2$?

3 Working in Quadrant I

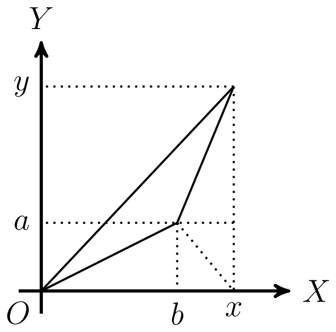
Now that I have some interest to know what all possible cases are for this problem, I would start with observing what are the possibilities when the triangle is entirely located in Quadrant I; that is, when all the coordinates x, y, b and a are positive. Compared with only looking at some arbitrary cases and being content, performing a thorough case analysis within Quadrant I is a step forward. Compared with trying to list all location possibilities across the Cartesian plane at once, to confine ourselves within Quadrant I is to take a *conservative* step forward, contributing towards our insight of the problem without overstretching our imagination.

3.1 Discovering the Cases

The experience of studying the two cases suggests that we can discover more cases by comparing between the X-coordinates (x and b) and between the Y-coordinates (y and a). The result of such comparisons determines how to express the heights or sides of triangles in terms of the coordinates.

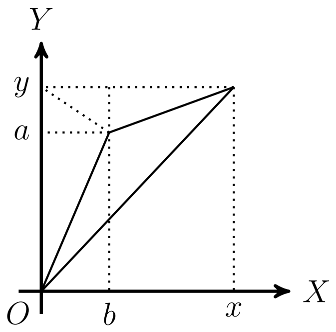
For example, only after assuming that the larger of the X-coordinates is x, and that the larger of the Y-coordinates is a, can we produce Figure 1 and determine that the lengths of the legs of the right triangle on the top-right corner of the rectangle shall be expressed by $x - b$ and $a - y$ (or by their equals) respectively, but not by $b - x$ nor $y - a$ (nor their equals).

The cases so discovered are summarized by Table 1. Cases 1 and 4 are illustrated by Figures 3 and 4 respectively. Case 2 is depicted by Figure 1. We see that a case may have sub-cases.



$$xy/2 - ax/2 - (x-b)y/2 = (by - ax)/2 \tag{4}$$

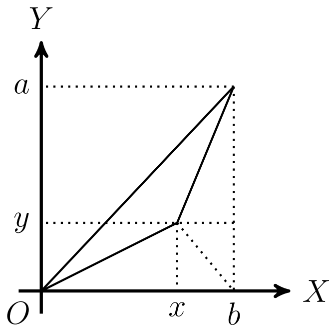
(a) The vertex (b, a) locates in the lower half of the rectangle.



$$xy/2 - (y-a)x/2 - by/2 = (ax - by)/2 \tag{5}$$

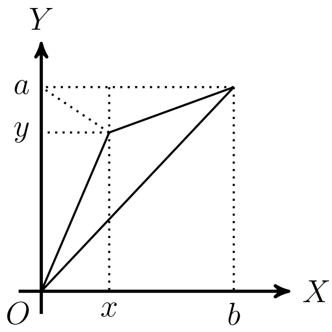
(b) The vertex (b, a) locates in the upper half of the rectangle.

Figure 3: The vertices (b, a) and (x, y) are both in Quadrant I, with $b < x$ and $a < y$.



$$ab/2 - by/2 - a(b - x)/2 = (ax - by)/2 \tag{6}$$

(a) The vertex (x, y) locates in the lower half of the rectangle.



$$ab/2 - ax/2 - b(a - y)/2 = (by - ax)/2 \tag{7}$$

(b) The vertex (x, y) locates in the upper half of the rectangle.

Figure 4: The vertices (b, a) and (x, y) are both in Quadrant I, with $b > x$ and $a > y$.

3.2 Coordinate-Swapping Symmetry

Observe the symmetry between Figure 3a and 4a: swapping x with b, and a with y, we can obtain one figure from the other.[1] Also observe that swappiing coordinate symbols like this has the effect of swapping the terms in the subtractions: $by - ax$ becomes $ax - by$ and vice versa.

If we know the formula for a case, then we can apply coordinate-swapping symmetry to derive straightly the formula for its opposite case without resorting to the algorithm of subtracting triangles from a rectangle. For example in Figure 1 the area is $(ax - by)/2$, then for its opposite (case 3 of Table 1) we can immediately write the formula $(by - ax)/2$ by swapping the terms ax and by.

3.3 What's next ?

Using symmetry we only need to run tediously the algorithm on some of the cases to get their formulae, from which the formulae for the rest can be derived. Now we ask: What is the full picture for the possible cases of the problem? Which of the cases must be approached with the algorithm, and which can be resolved by symmetry ? How much can symmetry help reducing the number of cases that require a "hard approach"?

4 The Landscape

Having gained enough of experience with the details of the problem, we are now well positioned to deliver the strongest blow, which shall exhibit the full extent of the problem. A vertex can be within a quadrant or on an axis but not on the origin. The two vertices cannot both be on the same axis. There are four quadrants and each of the two axes with the origin removed has a positive half and a negative half. Therefore each vertex has eight possible locations. Then the configuration of two vertices has 8×8 broad cases, which we organize by Table 2.

4.1 Case Reduction

Table 2 shows 56 (that is $8 \times 8 - 8$) possible cases and each case many have some sub-cases — we have already seen the sub-cases when both vertices are in Quadrant I. We value hard work and would appreciate the effort if

[1] The symmetry is also seen between Figure 3b and 4b.

(b,a) \ (x,y)	I	II	III	IV	X^+	X^-	Y^+	Y^-
I	◯	◯	◯	◯	◯	◯	◯	◯
II	◯	◯	◯	◯	◯	◯	◯	◯
III	◯	◯	◯	◯	◯	◯	◯	◯
IV	◯	◯	◯	◯	◯	◯	◯	◯
X^+	◯	◯	◯	◯	$\times$	$\times$	◯	◯
X^-	◯	◯	◯	◯	$\times$	$\times$	◯	◯
Y^+	◯	◯	◯	◯	◯	◯	$\times$	$\times$
Y^-	◯	◯	◯	◯	◯	◯	$\times$	$\times$

Table 2: All possible configurations of the two vertices on the Cartesian plane. A circle (◯) denotes a possible case (which may admit sub-cases) and a cross ($\times$) denotes an invalid case.

someone takes the time to go through the 56 cases and apply the rectangle-minus-triangle algorithm individually to each case and sub-case. But we also advocate insight and deeper understanding, if he could additionally realize that the cases are symmetric and by studying this symmetry he could reduce his workload by approximately (and as we shall see) 89%.

4.1.1 By Coordinate-Swapping Symmetry

The case where (b, a) is in Quadrant III and (x, y) is in Quadrant IV, and the case where (b, a) is in Quadrant IV and (x, y) is in Quadrant III (abbreviated case III-IV and case IV-III respectively) are opposites with respect to coordinate-swapping symmetry. Similarly, in Table 2, any pair of cases that is symmetric with respect to the (top-left to bottom-right) diagonal enjoys this symmetry.[2] That is, the formula(e) for case i-j ($i, j \in \{I, II, III, IV, X^+, X^-, Y^+, Y^-\}$ and $i \neq j$) differs from the formula(e) for case j-i only by a swapping of the symbols. Therefore if we know, say, the formulae for cases in the lower half of the case table, we can immediately write down those for the upper half by rewriting the symbols. Coordinate-swapping symmetry reduces our workload by 46% (the upper half, 26 circles in the case table) and more, since it may exist between sub-cases as per our study of Quadrant I.

4.1.2 By Mirror Symmetry and Rotational Symmetry

Within the lower half of the case table, we can further reduce the cases by mirror symmetry (wrt. the X and Y axes) and rotational symmetry (rotation by multiples of 90 degrees). For example, case I-I and case II-II are symmetric wrt. the Y-axis, so that if $f(b, a, x, y)$ is the formula for case I-I and $f'(b, a, x, y)$ is for case II-II, then we can relate the two formulae by

$$f'(b, a, x, y) = f(-b, a, -x, y).$$

If we know the concrete form of f then we can derive f' using the equation, and vice versa. Such connections among the cases are depicted in Table 3. If we regard the set of circles and edges in the table as a graph, there are six connected sub-graphs, forming some kind of equivalence clasess. Further, if we have concrete formula(e) for just one case in each of the six classes, formulae for all the rest cases in the highlghted area can be derived, and

[2] The terms *symmetry* and *symmetric* may seem a bit entangled here. We refer to the goemetric symmetry of the circles in Table 2 wrt. to the table's diagonal. Moreover, we refer to the symbolic symmetry between cases of the problem. The table is designed to indicate the latter by the former.

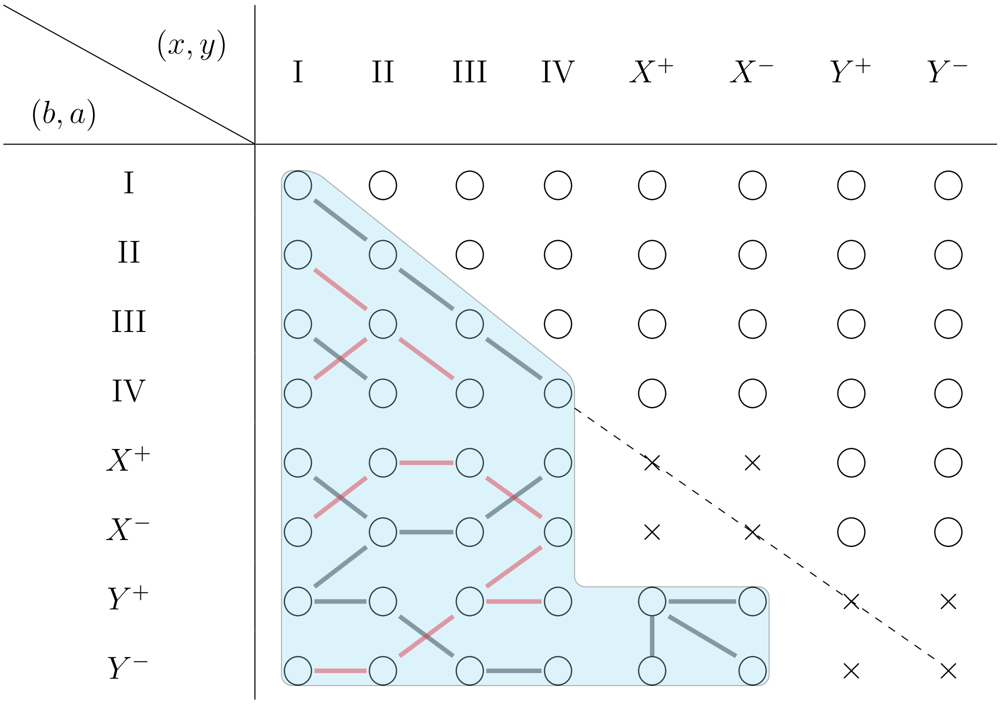


Table 3: A scheme for case reduction. Cases connected by edges enjoy some symmetry. The colors of the edges are only for readability of connected sub-graphs.

then by coordinate-swapping symmetry we derive the formulae for the upper half of the table.

5 Summary

Facing this problem (Section 1) I started with two arbitrary cases, observing that their formulae have opposite signs (Section 2).

I then moved on from the arbitrary cases to study systematically the situation where all coordinates are positive. I discovered an important kind of symmetry that can be used to reduce half of the cases to the other half, so that the formulae for the former can be derived from those of the latter by simply swapping the coordinate symbols.

Having discovered coordinate-swapping symmetry, I decided to have a systematic study of all possible cases. A case is characterized by the sign and zeroness combination of the coordinates.There are a total of 56 cases (Table 2) and in each inhabits one or more sub-cases determined by the relative magnitude of the coordinates.

After I identified all cases, I did not immediately dive into an exhaustive examination, because there are too many cases and I have already seen the application of coordinate-swapping symmetry in reducing the number of cases that have to be "hard calculated". Instead I systematically applied coordinate-swapping symmetry to eliminate all cases whose formulae can be derived from those of related cases. Further I used mirror and rotational symmetry to establish connections among the formulae for the remaining cases. This further reduces the number of "hard calculated" cases.

The key techniques involved in the proof are:

- Case Organization:
  - Level one: by sign and zeroness of the coordinates.
  - Level two: by relative magnitude of the coordinates.
- Case Reduction: finding and applying symmetries.

Although by now the proof is still not completed, there is nothing left except some hard work[3], which I would rather save for other endevours since I am quite satisfied with the more matured approach to the problem which I just took, compared with what I would do when I was a secondary school student[4].

[3]See also the beginning of Secion 4.1

[4]See also Section 2.2.